Supplement to Chapter 17: Dynamic Systems Modelling

Section 17.3 Two Coupled Systems

Exercise, page 147.

For this exercise the dimensions and parameters to be used for the two tanks are: $C = 0.001\ m^2/s$, $A = 10\ m^2$, $\Delta t = 1$ hour (3600 s).

In your numerical method use the simplest approximations to the mean values of $\bar{x}_1$ and $\bar{x}_2$ as follows: $\bar{x}_1 = x_1(t_i)$, $\bar{x}_2 = x_2(t_i)$.

Advection

Advection is the process by which properties of the atmosphere are carried by the wind. If, for example, the air is colder in the west than it is in the east, then a wind from the west will cause the temperature at a fixed position to fall.

A simple model of this process is the *advection equation*:

$$\partial T(x,t)/\partial t = -u \partial T(x,t)/\partial x,$$

where $T(x,t)$ is the temperature at position x and time t, and u is the wind velocity in the x-direction. In the following examples we assume the wind velocity u is constant.

To obtain a numerical solution of this equation we may use a simple *upwind approximation* as follows. Divide the time into equal intervals Δt, and divide the path of the wind into equal steps Δx. When u is positive we write the numerical approximation as:

$$[T(x, t + \Delta t) - T(x, t)]/\Delta t$$
$$= -u[T(x, t) - T(x - \Delta x, t)]/\Delta x,$$

and when u is negative we write:

$$[T(x, t + \Delta t) - T(x, t)]/\Delta t$$
$$= -u[T(x + \Delta x, t) - T(x, t)]/\Delta x.$$

Example

Let u = 10 m/s (36 km/hour) and Δx = 1 km. Suppose the temperature distribution over a distance 10 km has a wave as shown in the following table when t = 0 s. The temperatures at later times calculated by the advection model depend on the time step Δt used.

Initial Temperatures (°C)

x (km)	0	1	2	3	4	5	6	7	8	9	10
T (°C)	20	20	20	21	24	25	24	21	20	20	20

(a) Let Δt = 50 s. Then the upwind approximation is

$$T(x, t + \Delta t) = 0.5T(x - \Delta x, t) + 0.5T(x, t).$$

Calculations using this approximation shift the temperature wave forward in the x-direction as shown in Fig. 1, but the height of the wave is reduced and the width of the wave is increased. This is an unreal dissipation effect caused by the numerical method because the time step Δt is less than the time it takes the wind to cover the distance Δx.

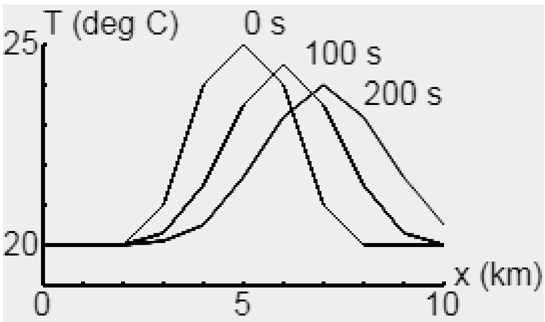

Fig. 1. Results using Δt = 50 s.

(b) Let Δt = 100 s. Then the upwind approximation is

$$T(x, t + \Delta t) = T(x - \Delta x, t).$$

Using this approximation the calculated temperature wave is shifted forward in the x-direction as shown in Fig. 2 without any change in the shape of the wave. This is because $\Delta x/\Delta t$ is equal to the wind speed u.

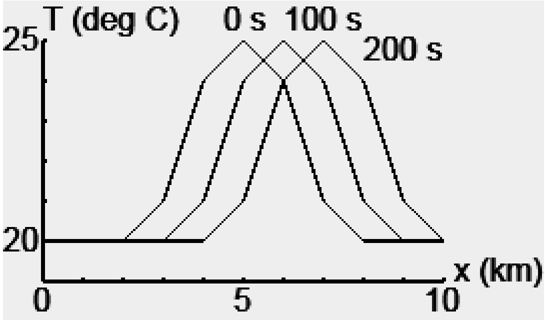

Fig. 2. Results using Δt = 100 s.

(c) Let Δt = 200 s. Then the upwind approximation is

$$T(x, t + \Delta t) = 2T(x - \Delta x, t) - T(x, t).$$

Figure 3 shows that in this case the model fails to represent the advection of the temperature wave correctly. After more steps a numerical overflow will cause the model to crash. This is because the time step Δt is greater than the time it takes the wind to cover the distance Δx.

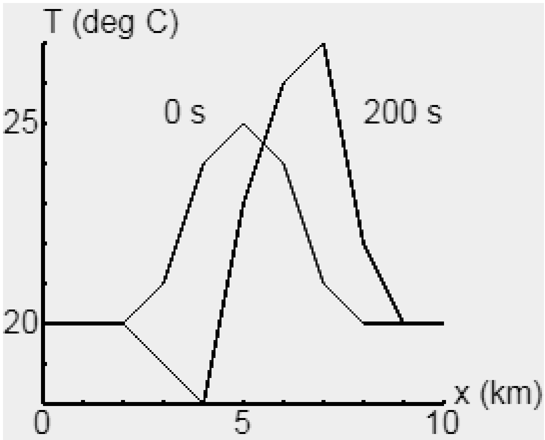

Fig. 3. Results using Δt = 200 s.

Exercise

Calculate the advection of the wave in the table above with the wind blowing in the opposite direction ($u = -10$ m/s). Use the upwind approximation equation for negative values of u. Do two calculations: one with Δt = 60 s (one minute) and the other with Δt = 120 s (two minutes). Write comments on your results.

17.5.2 The Verhulst Population Model

Do the exercise on page 154.
